Maximise area given perimeter

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Abstract

It is not at all surprising that a regular polygon has a larger area for a given perimiter than any other polygon with the same number of sides, but can you prove it? It is remarkably hard. This question was proposed for a GCSE maths project but I can't find any proof that does not involve some concepts from calculus and linear algebra, albeit disguised. Here's my best effort.

1 Triangles

The case of a triangle is easy. Given any triangle ABC, fix A and B and try moving C parallel to the line AB. This changes the perimeter but not the area of the triangle. It is not hard to see that the perimiter is minimised when the triangle is isosceles. We can also move A parallel to the line BC, or B parallel to the line AC, so to minimise the perimeter for a given area the triangle must be isosceles in three different ways, that is, equilateral.

We should pause to prove that minimising the perimeter for a given area is the same as maximising the area for a given perimeter. Suppose that a polygon X has maximal area A for a given perimeter p and that a second polygon Y with the same number of sides has the same area but a smaller perimeter q. Construct a third polygon Z by enlarging Y by a factor of p/q. Its perimeter is now equal to p but its area is $A\left(\frac{p}{q}\right)^2$ which is larger than A. This contradicts our assumption that X has maximal area. Therefore our two assumptions cannot simultaneously hold. Therefore if A is the maximal area for perimeter p then p must be the minimal perimeter for area A. Exchange X and Y to prove the converse.

2 Quadralaterals

Now consider a quadralateral $ABCD$. We can cut it into two triangles ABC and CDA. By moving B parallel to AC we can show that ABC must be isosceles, that is that $AB = BC$. Similarly we can show that $BC = CD$ and $CD = DA$ so all the sides must be the same length. This tells us that the quadralateral must be a rhombus, but we still haven't proved that it must be a square.

Fix AB to be horizontal and choose the angle ABC. This fixes C somewhere on the circle with centre B and radius BA. That in turn fixes D because CD must be parallel and equal to BA. The area of our rhombus is the length of the base BA times the perpendicular height of C above BA. Now keep the base fixed and vary the angle ABC so as to maximise the height. Clearly the maximum occurs when ABC is a right-angle, that is when $ABCD$ is a square.

3 Cannot generalise

The first half of the argument for quadralaterals generalises to polygons with any number of sides. Consider a polygon $ABC...Z$. Cut off a triangle ABC and move B parallel to AC. The perimeter is minimised for a given area when $AB = BC$. Since we could have chosen any corner to move instead of B, all sides must be the same length.

The second half of the argument for quadralaterals is specific to quadralaterals and does not generalise. We've proved that the optimal polygon has all its sides equal, but we need to prove that it has all its angles equal too. That's the hard part.

4 Exchange rates

Again consider a polygon $ABC...Z$ and cut off a triangle ABC. Let us suppose we have already arranged for all the sides to be equal so that $AB = BC$. Now consider moving B perpendicular (not parallel) to AC by an infinitesimal amount dh. This changes both the area and the perimeter, so at first sight we're on to a loser.

However, the change of area dA and the change of perimeter dp are both proportional to dh (to first order) and so we can work out the ratio dA/dp, and call it k. Given that we have assumed that ABC is isosceles, k can only depend on one thing: the angle ABC. Even without computing it explicitly, it is not hard to see that k is an increasing function of the angle ABC, equal to infinity when ABC is 180 degrees (that is, when ABC is a straight line) and decreasing to a limit of $\frac{1}{2}AC$ as ABC decreases to zero (that is, when ABC is very acute).

This ratio k is like an exchange rate of perimeter for area.[1] We computed it for the corner B but we could have chosen any corner. Other corners might have

[1] Technically, it is a Lagrange multiplier.

different exchange rates. If so, there is an economic short-circuit, and we can exploit it in order to increase the area without changing the perimeter, or to decrease the perimeter without changing the area. That makes the basis of a proof.

5 The proof

Suppose a polygon $ABC...Z$ maximises the area for a given perimeter relative to other polygons with the same number of sides. Obviously it is convex. By the argument in section 3, we know that all of its sides are the same length. Let us suppose, hoping for a contradiction, that two of its angles are different. For the sake of the argument, let's say they are the angles at B and E, but they could be at any two corners, even corners that are next to each other. Without loss of generality, let's suppose that angle ABC is smaller (more acute) than angle DEF.

Compute the exchange rate k_B for moving B an infinitesimal distance perpendicular to AC, and also the exchange rate k_E for moving E perpendicular to DF. Because ABC is smaller than DEF, we know k_B is smaller than k_E.

Choose an infinitesimal change of perimeter dp. Move B perpendicular to AC towards the centre of the polygon just enough to decrease the perimeter by dp. This also decreases the area by $k_B dp$. Now move E perpendicular to DF away from the centre of the polygon just enough to increase the perimeter by dp. This increases the area by $k_E dp$.

The total change of perimeter is zero, but the total change of area is $(k_E - k_B)dp$ which is greater than zero. This contradicts our assumption that the polygon was optimal. Therefore, if the polygon is optimal then it must be impossible to find two angles that are different. Since we already know that all the sides must be equal, this proves that the polygon is regular.

6 Corollary

By considering the limit as the number of sides tends to infinity, we have proved that a circle has a larger area for a given perimeter than any other closed curve.